

On the cardinality of general h -fold sumsets ^{*}

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Abstract

Let $A = \{a_0, a_1, \dots, a_{k-1}\}$ be a set of k integers. For any integer $h \geq 1$ and any ordered k -tuple of positive integers $\mathbf{r} = (r_0, r_1, \dots, r_{k-1})$, we define a general h -fold sumset, denoted by $h^{(\mathbf{r})}A$, which is the set of all sums of h elements of A , where a_i appearing in the sum can be repeated at most r_i times for $i = 0, 1, \dots, k-1$. In this paper, we give the best lower bound for $|h^{(\mathbf{r})}A|$ in terms of \mathbf{r} and h and determine the structure of the set A when $|h^{(\mathbf{r})}A|$ is minimal. This generalizes results of Nathanson, and recent results of Mistri and Pandey and also solves a problem of Mistri and Pandey.

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1 Introduction

Let \mathbb{N} denote the set of all nonnegative integers. For any finite set of integers A and any positive integer $h \geq 2$, define

$$hA = \{a_1 + a_2 + \cdots + a_h : a_i \in A(1 \leq i \leq h)\}$$

and

$$h\hat{A} = \{a_1 + a_2 + \cdots + a_h : a_i \in A(1 \leq i \leq h), a_i \neq a_j \text{ for all } i \neq j\}.$$

Sumsets are important in additive number theory (see [1–3, 5, 8–11]).

Finding lower bounds for $|hA|$ and $|h\hat{A}|$ in terms of h and $|A|$ and determining the structure of sets A for which $|hA|$ or $|h\hat{A}|$ are minimal are important problems in additive number theory.

Nathanson [7] proved the following fundamental and important results.

Theorem A. (See [7, Theorem 1.3]) *Let $h \geq 2$ be an integer and A a finite set of integers with $|A| = k$. Then*

$$|hA| \geq hk - h + 1.$$

Theorem B. (See [7, Theorem 1.6]) *Let $h \geq 2$ be an integer and A a finite set of integers with $|A| = k$. Then*

$$|hA| = hk - h + 1$$

if and only if A is a k -term arithmetic progression.

Theorem C. (See [7, Theorem 1.9] or [6, Theorem 1]) *Let A be a finite set of integers with $|A| = k$ and let $1 \leq h \leq k$. Then*

$$|h\hat{A}| \geq hk - h^2 + 1.$$

This lower bound is best possible.

Theorem D. (See [7, Theorem 1.10] or [6, Theorem 2]) *Let $k \geq 5$ and let $2 \leq h \leq k - 2$. If A is a set of k integers such that*

$$|h\hat{A}| = hk - h^2 + 1,$$

then A is a k -term arithmetic progression.

From now on, we assume that $A = \{a_0, a_1, \dots, a_{k-1}\}$ is a set of integers with $a_0 < a_1 < \dots < a_{k-1}$. For two positive integers h and r , define

$$h^{(r)}A = \left\{ \sum_{i=0}^{k-1} s_i a_i : 0 \leq s_i \leq r \text{ for } i = 0, 1, \dots, k-1 \text{ and } \sum_{i=0}^{k-1} s_i = h \right\}.$$

Clearly, $h^{(1)}A = \hat{h}A$ and $h^{(h)}A = hA$. Recently, Mistri and Pandey generalized the above results.

Theorem E. (See [4, Theorem 2.1]) *Let A be a set of k integers, r and h be two integers such that $1 \leq r \leq h \leq rk$. Then*

$$|h^{(r)}A| \geq mr(k-m) + (h-mr)(k-2m-1) + 1,$$

where m is the integer with $h/r - 1 < m \leq h/r$. This lower bound is best possible.

Theorem F. (See [4, Theorem 3.1, Theorem 3.2]) *Let $k \geq 3$, r and h be integers with $1 \leq r \leq h \leq rk - 2$ and $(k, h, r) \neq (4, 2, 1)$. If A is a set of k integers such that*

$$|h^{(r)}A| = mr(k-m) + (h-mr)(k-2m-1) + 1,$$

where m is the integer with $h/r - 1 < m \leq h/r$, then A is a k -term arithmetic progression.

For any ordered k -tuple of positive integers $\mathbf{r} = (r_0, r_1, \dots, r_{k-1})$ and any positive integer h , define

$$h^{(\mathbf{r})}A = \left\{ \sum_{i=0}^{k-1} s_i a_i : 0 \leq s_i \leq r_i (0 \leq i \leq k-1), \sum_{i=0}^{k-1} s_i = h \right\}.$$

Clearly, if $\mathbf{r} = (r, r, \dots, r)$ is an ordered k -tuple of positive integers, then $h^{(\mathbf{r})}A = h^{(r)}A$.

Mistri and Pandey [4, Concluding Remarks] said that it is interesting to study the direct and inverse problems related to sumset $h^{(\mathbf{r})}A$.

In this paper, we solve this problem.

For convenience, let $\sum_{x=a}^b f(x) = 0$ if $a > b$. Let $I_{\mathbf{r}}(h)$ be the largest integer and $M_{\mathbf{r}}(h)$ be the least integer such that

$$\sum_{j=0}^{I_{\mathbf{r}}(h)-1} r_j \leq h, \quad \sum_{j=M_{\mathbf{r}}(h)+1}^{k-1} r_j \leq h,$$

and let

$$\delta_{\mathbf{r}}(h) = h - \sum_{j=0}^{I_{\mathbf{r}}(h)-1} r_j, \quad \theta_{\mathbf{r}}(h) = h - \sum_{j=M_{\mathbf{r}}(h)+1}^{k-1} r_j.$$

Let

$$L(\mathbf{r}, h) = \sum_{j=M_{\mathbf{r}}(h)+1}^{k-1} jr_j - \sum_{j=0}^{I_{\mathbf{r}}(h)-1} jr_j + M_{\mathbf{r}}(h)\theta_{\mathbf{r}}(h) - I_{\mathbf{r}}(h)\delta_{\mathbf{r}}(h) + 1.$$

In this paper, we prove the following theorems.

Theorem 1.1. *Let $A = \{a_0, a_1, \dots, a_{k-1}\}$ be a set of integers with $a_0 < a_1 < \dots < a_{k-1}$, $\mathbf{r} = (r_0, r_1, \dots, r_{k-1})$ be an ordered k -tuple of positive integers and h be an integer with*

$$2 \leq h \leq \sum_{j=0}^{k-1} r_j.$$

Then

$$|h^{(\mathbf{r})}A| \geq L(\mathbf{r}, h).$$

This lower bound is best possible.

Theorem 1.2. *Let $k \geq 5$ be an integer, $\mathbf{r} = (r_0, r_1, \dots, r_{k-1})$ be an ordered k -tuple of positive integers and let h be an integer with*

$$2 \leq h \leq \sum_{j=0}^{k-1} r_j - 2.$$

If A is a set of k integers, then

$$|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$$

if and only if A is a k -term arithmetic progression.

Remark 1.1. For Theorem 1.2 with $1 \leq k \leq 4$, we shall give complete results in Section 3. Since

$$L((r, r, \dots, r), h) = mr(k - m) + (h - mr)(k - 2m - 1) + 1,$$

Theorem F is a corollary of Theorem 1.2 and Theorems 3.1 and 3.2 in Section 3.

Remark 1.2. If $h = 1$, then $h^{(\mathbf{r})}A = A$. So $|h^{(\mathbf{r})}A| = k$.

If

$$h = \sum_{j=0}^{k-1} r_j - 1,$$

then

$$h^{(\mathbf{r})}A = \left\{ \sum_{j=0}^{k-1} r_j a_j - a_i : 0 \leq i \leq k-1 \right\}.$$

So $|h^{(\mathbf{r})}A| = k$.

If

$$h = \sum_{j=0}^{k-1} r_j,$$

then

$$h^{(\mathbf{r})}A = \left\{ \sum_{j=0}^{k-1} r_j a_j \right\}.$$

So $|h^{(\mathbf{r})}A| = 1$.

2 Proofs

For any k -tuple $X = (x_0, x_1, \dots, x_{k-1}) \in \mathbb{N}^k$, define the function

$$\phi_A(X) = \sum_{j=0}^{k-1} x_j a_j.$$

For any ordered k -tuple of positive integers $\mathbf{r} = (r_0, r_1, \dots, r_{k-1})$ and any positive integer h , let $R(\mathbf{r}, h)$ be the set of all ordered k -tuple $(x_0, x_1, \dots, x_{k-1})$ of \mathbb{N}^k such that

$$\sum_{j=0}^{k-1} x_j = h, \quad 0 \leq x_i \leq r_i, \quad i = 0, 1, \dots, k-1.$$

Then

$$h^{(\mathbf{r})}A = \{\phi_A(X) : X \in R(\mathbf{r}, h)\}.$$

For any positive integer k and any k -tuple $X = (x_0, x_1, \dots, x_{k-1}) \in \mathbb{N}^k$, define the weighted sum

$$S(X) = \sum_{j=0}^{k-1} jx_j.$$

For two k -tuples $U = (u_0, u_1, \dots, u_{k-1}), W = (w_0, w_1, \dots, w_{k-1}) \in \mathbb{N}^k$, we call $U \rightarrow W$ a *step* if there exists an index $j \geq 0$ such that $w_j = u_j - 1$, $w_{j+1} = u_{j+1} + 1$ and $w_i = u_i$ for all integers $i \neq j, j+1$. We call $X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_t$ a (\mathbf{r}, h) -*path* of length t , if $X_i \in R(\mathbf{r}, h) (1 \leq i \leq t)$ and $X_{i+1} \rightarrow X_i (1 \leq i \leq t-1)$ are steps. It is clear that if $X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_t$ is a (\mathbf{r}, h) -path of length t , then

$$S(X_{i+1}) - S(X_i) = 1 (1 \leq i \leq t-1).$$

Thus $S(X_t) - S(X_1) = t - 1$.

Let

$$V = (r_0, r_1, \dots, r_{I_{\mathbf{r}}(h)-1}, \delta_{\mathbf{r}(h)}, 0, \dots, 0)$$

and

$$V' = (0, \dots, 0, \theta_{\mathbf{r}(h)}, r_{M_{\mathbf{r}}(h)+1}, \dots, r_{k-1}),$$

where $I_{\mathbf{r}}(h), \delta_{\mathbf{r}(h)}, \theta_{\mathbf{r}(h)}, M_{\mathbf{r}}(h)$ are defined as in Section 1. Then $V, V' \in R(\mathbf{r}, h)$.

Lemma 2.1. *We have $S(V') - S(V) + 1 = L(\mathbf{r}, h)$. In particular, any (\mathbf{r}, h) -path from V to V' has length $L(\mathbf{r}, h)$.*

Proof. Noting that

$$S(V) = \sum_{j=0}^{I_{\mathbf{r}}(h)-1} jr_j + I_{\mathbf{r}}(h)\delta_{\mathbf{r}(h)}, \quad S(V') = \sum_{j=M_{\mathbf{r}}(h)+1}^{k-1} jr_j + M_{\mathbf{r}}(h)\theta_{\mathbf{r}(h)},$$

we have

$$\begin{aligned} S(V') - S(V) &= \sum_{j=M_{\mathbf{r}}(h)+1}^{k-1} jr_j - \sum_{j=0}^{I_{\mathbf{r}}(h)-1} jr_j + M_{\mathbf{r}}(h)\theta_{\mathbf{r}(h)} - I_{\mathbf{r}}(h)\delta_{\mathbf{r}(h)} \\ &= L(\mathbf{r}, h) - 1. \end{aligned}$$

Since a (\mathbf{r}, h) -path from V to V' has length $S(V') - S(V) + 1$, it follows that any (\mathbf{r}, h) -path from V to V' has length $L(\mathbf{r}, h)$. \square

Lemma 2.2. *Let $X = (x_0, x_1, \dots, x_{k-1}) \in R(\mathbf{r}, h)$ and $Y = (y_0, y_1, \dots, y_{k-1}) \in R(\mathbf{r}, h)$ with $X \neq Y$. If*

$$\sum_{j=i}^{k-1} x_j \leq \sum_{j=i}^{k-1} y_j, \quad i = 1, 2, \dots, k-1,$$

then there exists a (\mathbf{r}, h) -path from X to Y .

Proof. Let $X_0 = X \rightarrow X_1 \rightarrow \dots \rightarrow X_g$ be a (\mathbf{r}, h) -path of the maximal length such that

$$(1) \quad \sum_{j=t}^{k-1} x_{i,j} \leq \sum_{j=t}^{k-1} y_j, \quad 1 \leq t \leq k-1, 1 \leq i \leq g,$$

where $X_i = (x_{i,0}, x_{i,1}, \dots, x_{i,k-1})$ ($0 \leq i \leq g$). Now we prove that $X_g = Y$. Suppose that $X_g \neq Y$. Let s be the maximal index with $x_{g,s} \neq y_s$. Noting that $X, Y \in R(\mathbf{r}, h)$, we have

$$\sum_{j=0}^{k-1} x_{g,j} = h = \sum_{j=0}^{k-1} y_j.$$

Hence $s \geq 1$. Since

$$\sum_{j=s}^{k-1} x_{g,j} \leq \sum_{j=s}^{k-1} y_j,$$

it follows from the definition of s that $x_{g,s} < y_s$. If $x_{g,s-1} > 0$, let

$$X_{g+1} = (x_{g,0}, \dots, x_{g,s-1} - 1, x_{g,s} + 1, x_{g,s+1}, \dots, x_{g,k-1}),$$

then $X_g \rightarrow X_{g+1}$ is a (\mathbf{r}, h) -path and X_{g+1} also satisfies (1). This is a contradiction with the maximality of g . Hence $x_{g,s-1} = 0$. If $x_{g,j} = 0$ for all

$0 \leq j \leq s-1$, then

$$\begin{aligned}
\sum_{j=0}^{k-1} x_{g,j} &= x_{g,s} + \sum_{j=s+1}^{k-1} x_{g,j} \\
&= x_{g,s} + \sum_{j=s+1}^{k-1} y_j \\
&< y_s + \sum_{j=s+1}^{k-1} y_j \\
&\leq \sum_{j=0}^{k-1} y_j = h,
\end{aligned}$$

a contradiction with $X_g \in R(\mathbf{r}, h)$ (see the definition of (\mathbf{r}, h) -path). Thus there exists an index j with $0 \leq j < s-1$ such that $x_{g,j} > 0$. We assume that j is the largest such index. Let

$$X_{g+1} = (x_{g,0}, \dots, x_{g,j} - 1, x_{g,j+1} + 1, 0, \dots, 0, x_{g,s}, \dots, x_{g,k-1}).$$

Then $X_g \rightarrow X_{g+1}$ is a (\mathbf{r}, h) -path. Since X_g satisfies (1), it follows that X_{g+1} also satisfies (1). This is a contradiction with the maximality of g . Therefore, $X_g = Y$. \square

Lemma 2.3. *Let $X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_{t-1} \rightarrow X_t$ and $X_1 \rightarrow X'_2 \rightarrow \dots \rightarrow X'_{t-1} \rightarrow X_t$ be two different (\mathbf{r}, h) -paths from X_1 to X_t . If A is a set of k integers such that $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$, then $\phi_A(X_i) = \phi_A(X'_i)$ for $i = 2, 3, \dots, t-1$.*

Proof. By Lemma 2.2, there exists a (\mathbf{r}, h) -path from V to X_1 and another (\mathbf{r}, h) -path from X_t to V' . Thus we have the following (\mathbf{r}, h) -path from V to V' :

$$(2) \quad V \rightarrow \dots \rightarrow X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_{t-1} \rightarrow X_t \rightarrow \dots \rightarrow V'.$$

By Lemma 2.1, the length of the (\mathbf{r}, h) -path (2) is $L(\mathbf{r}, h) = |h^{(\mathbf{r})}A|$. Clearly,

$$\phi_A(V) < \dots < \phi_A(X_1) < \phi_A(X_2) < \dots < \phi_A(X_{t-1}) < \phi_A(X_t) < \dots < \phi_A(V').$$

Since

$$\{\phi_A(X) : X \text{ is on the } (\mathbf{r}, h)\text{-path (2)}\} \subseteq h^{(\mathbf{r})}A$$

and

$$|\{\phi_A(X) : X \text{ is on the } (\mathbf{r}, h)\text{-path } (2)\}| = |h^{(\mathbf{r})}A|,$$

it follows that

$$h^{(\mathbf{r})}A = \{\phi_A(X) : X \text{ is on the } (\mathbf{r}, h)\text{-path } (2)\}.$$

Noting that

$$\{\phi_A(X'_2), \phi_A(X'_3), \dots, \phi_A(X'_{t-1})\} \subseteq h^{(\mathbf{r})}A$$

and

$$\phi_A(X_1) < \phi_A(X'_2) < \dots < \phi_A(X'_{t-1}) < \phi_A(X_t),$$

we have $\phi_A(X_i) = \phi_A(X'_i)$ for $i = 2, 3, \dots, t-1$. \square

Lemma 2.4. *Let c_i and d_i ($0 \leq i \leq k-1$) be integers with $c_i \leq d_i$ ($0 \leq i \leq k-1$). If h is an integer with*

$$\sum_{i=0}^{k-1} c_i \leq h \leq \sum_{i=0}^{k-1} d_i,$$

then there exist integers x_i ($0 \leq i \leq k-1$) with $c_i \leq x_i \leq d_i$ ($0 \leq i \leq k-1$) such that

$$h = x_0 + x_1 + \dots + x_{k-1}.$$

Proof is left to the reader.

Proof of Theorem 1.1. By Lemma 2.2, there exists a (\mathbf{r}, h) -path $V = V_0 \rightarrow V_1 \rightarrow \dots \rightarrow V_\ell = V'$. By Lemma 2.1, we have $\ell + 1 = L(\mathbf{r}, h)$. Since $\phi_A(V_i) \in h^{(\mathbf{r})}A$ ($0 \leq i \leq \ell$) and $\phi_A(V_{i+1}) > \phi_A(V_i)$ ($0 \leq i \leq \ell-1$), we have

$$(3) \quad |h^{(\mathbf{r})}A| \geq \ell + 1 = L(\mathbf{r}, h).$$

Next we show that this lower bound is optimal. Let $A = \{0, 1, \dots, k-1\}$. Then the smallest integer in $h^{(\mathbf{r})}A$ is

$$\begin{aligned} & \underbrace{0 + \dots + 0}_{r_0 \text{ copies}} + \underbrace{1 + \dots + 1}_{r_1 \text{ copies}} + \dots + \underbrace{(I_{\mathbf{r}}(h) - 1) + \dots + (I_{\mathbf{r}}(h) - 1)}_{r_{I_{\mathbf{r}}(h)-1} \text{ copies}} \\ & + \underbrace{I_{\mathbf{r}}(h) + \dots + I_{\mathbf{r}}(h)}_{\delta_{\mathbf{r}(h)} \text{ copies}} \\ = & S(V) \end{aligned}$$

and the largest integer in $h^{(\mathbf{r})}A$ is

$$\begin{aligned}
& \underbrace{M_{\mathbf{r}}(h) + \cdots + M_{\mathbf{r}}(h)}_{\theta_{\mathbf{r}}(h) \text{ copies}} + \underbrace{(M_{\mathbf{r}}(h) + 1) + \cdots + (M_{\mathbf{r}}(h) + 1)}_{r_{M_{\mathbf{r}}(h)+1} \text{ copies}} \\
& + \cdots + \underbrace{(k-2) + \cdots + (k-2)}_{r_{k-2} \text{ copies}} + \underbrace{(k-1) + \cdots + (k-1)}_{r_{k-1} \text{ copies}} \\
& = S(V').
\end{aligned}$$

It follows that

$$h^{(\mathbf{r})}A \subseteq [S(V), S(V')].$$

Thus, by Lemma 2.1, we have

$$(4) \quad |h^{(\mathbf{r})}A| \leq S(V') - S(V) + 1 = L(\mathbf{r}, h).$$

By (3) and (4), we have

$$|h^{(\mathbf{r})}A| = L(\mathbf{r}, h).$$

□

Proof of Theorem 1.2. Suppose that $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$. For any integer j with $0 \leq j \leq k-4$, by

$$2 \leq h \leq \sum_{i=0}^{k-1} r_i - 2$$

and Lemma 2.4, there exists

$$X = (x_0, x_1, \dots, x_j, x_{j+1}, x_{j+2}, x_{j+3}, \dots, x_{k-1}) \in R(\mathbf{r}, h)$$

such that

$$1 \leq x_j \leq r_j, \quad 0 \leq x_{j+1} \leq r_{j+1}-1, \quad 1 \leq x_{j+2} \leq r_{j+2}, \quad 0 \leq x_{j+3} \leq r_{j+3}-1.$$

Then

$$\begin{aligned}
& (\dots, x_j, x_{j+1}, x_{j+2}, x_{j+3}, \dots) \\
& \rightarrow (\dots, x_j - 1, x_{j+1} + 1, x_{j+2}, x_{j+3}, \dots) \\
& \rightarrow (\dots, x_j - 1, x_{j+1} + 1, x_{j+2} - 1, x_{j+3} + 1, \dots)
\end{aligned}$$

and

$$\begin{aligned}
& (\dots, x_j, x_{j+1}, x_{j+2}, x_{j+3}, \dots) \\
\rightarrow & (\dots, x_j, x_{j+1}, x_{j+2} - 1, x_{j+3} + 1, \dots) \\
\rightarrow & (\dots, x_j - 1, x_{j+1} + 1, x_{j+2} - 1, x_{j+3} + 1, \dots)
\end{aligned}$$

are two different (\mathbf{r}, h) -paths. By Lemma 2.3, we have

$$\phi_A((\dots, x_j - 1, x_{j+1} + 1, x_{j+2}, x_{j+3}, \dots)) = \phi_A((\dots, x_j, x_{j+1}, x_{j+2} - 1, x_{j+3} + 1, \dots)).$$

This implies that $a_{j+1} - a_j = a_{j+3} - a_{j+2}$. Therefore,

$$a_1 - a_0 = a_3 - a_2 = a_5 - a_4 = \dots, \quad a_2 - a_1 = a_4 - a_3 = a_6 - a_5 = \dots.$$

In order to prove that A is a k -term arithmetic progression, it suffices to prove $a_4 - a_3 = a_1 - a_0$.

By

$$2 \leq h \leq \sum_{i=0}^{k-1} r_i - 2$$

and Lemma 2.4, there exists

$$Y = (y_0, y_1, y_2, y_3, y_4, \dots, y_{k-1}) \in R(\mathbf{r}, h)$$

such that

$$1 \leq y_0 \leq r_0, \quad 0 \leq y_1 \leq r_1 - 1, \quad 1 \leq y_3 \leq r_3, \quad 0 \leq y_4 \leq r_4 - 1.$$

Then

$$\begin{aligned}
& (y_0, y_1, y_2, y_3, y_4, \dots, y_{k-1}) \\
\rightarrow & (y_0 - 1, y_1 + 1, y_2, y_3, y_4, \dots, y_{k-1}) \\
\rightarrow & (y_0 - 1, y_1 + 1, y_2, y_3 - 1, y_4 + 1, \dots, y_{k-1})
\end{aligned}$$

and

$$\begin{aligned}
& (y_0, y_1, y_2, y_3, y_4, \dots, y_{k-1}) \\
\rightarrow & (y_0, y_1, y_2, y_3 - 1, y_4 + 1, \dots, y_{k-1}) \\
\rightarrow & (y_0 - 1, y_1 + 1, y_2, y_3 - 1, y_4 + 1, \dots, y_{k-1})
\end{aligned}$$

are two different (\mathbf{r}, h) -paths. By Lemma 2.3, we have

$$\phi_A((y_0-1, y_1+1, y_2, y_3, y_4, \dots, y_{k-1})) = \phi_A((y_0, y_1, y_2, y_3-1, y_4+1, \dots, y_{k-1})).$$

This implies that $a_1 - a_0 = a_4 - a_3$.

Therefore, A is a k -term arithmetic progression.

Conversely, if A is a k -term arithmetic progression, without loss of generality, we may assume that $A = \{0, 1, \dots, k-1\}$. By the proof of Theorem 1.1, we have $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$. \square

3 Cases $1 \leq k \leq 4$

For $k = 1$ and $1 \leq h \leq r_0$, it is easy to see that $h^{(\mathbf{r})}A = \{ha_0\}$. So $|h^{(\mathbf{r})}A| = 1$.

For $k = 2$ and $1 \leq h \leq r_0 + r_1$, we have

$$h^{(\mathbf{r})}A = \{x_0a_0 + x_1a_1 : 0 \leq x_0 \leq r_0, 0 \leq x_1 \leq r_1, x_0 + x_1 = h, x_0, x_1 \in \mathbb{N}\}.$$

So

$$|h^{(\mathbf{r})}A| = |\{(x_0, x_1) : 0 \leq x_0 \leq r_0, 0 \leq x_1 \leq r_1, x_0 + x_1 = h, x_0, x_1 \in \mathbb{N}\}|.$$

Now we deal with the cases $k = 3$ and $k = 4$.

Theorem 3.1. *Let $A = \{a_0 < a_1 < a_2\}$ be a set of integers and $\mathbf{r} = (r_0, r_1, r_2)$ be an ordered 3-tuple of positive integers. Suppose that h is an integer with $2 \leq h \leq r_0 + r_1 + r_2 - 2$. Then*

(i) *for $r_1 = 1$, we have $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$;*

(ii) *for $r_1 \geq 2$, we have $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$ if and only if A is a 3-term arithmetic progression.*

Proof. We first prove (i). Suppose that $r_1 = 1$. By Lemma 2.2, there exists a (\mathbf{r}, h) -path from V to V' :

$$(5) \quad V = V_0 \rightarrow V_1 \rightarrow \dots \rightarrow V_t = V'.$$

Let $X = (x_0, x_1, x_2) \rightarrow Y$ be a (\mathbf{r}, h) -path. If $x_1 = 0$, then $Y = (x_0 - 1, 1, x_2)$. If $x_1 = 1$, then $Y = (x_0, 0, x_2 + 1)$. That is, Y is uniquely determined by X . Hence, the (\mathbf{r}, h) -path (5) is uniquely determined by V and V' . For any $W \in R(\mathbf{r}, h)$, by Lemma 2.2, there exists a (\mathbf{r}, h) -path from V to W and a (\mathbf{r}, h) -path W to V' . Since (5) is unique, we have $W \in \{V_0, V_1, \dots, V_t\}$. Thus, by the definition of $h^{(\mathbf{r})}A$, $\phi_A(V_i) < \phi_A(V_{i+1})$ ($0 \leq i \leq t - 1$) and Lemma 2.1, we have

$$\begin{aligned} |h^{(\mathbf{r})}A| &= |\{\phi_A(X) : X \in R(\mathbf{r}, h)\}| \\ &= |\{\phi_A(V_i) : i = 0, \dots, t\}| \\ &= t + 1 = S(V') - S(V) + 1 = L(\mathbf{r}, h). \end{aligned}$$

Next we shall prove (ii). If A is a 3-term arithmetic progression, without loss of generality, we may assume that $A = \{0, 1, 2\}$. By the proof of Theorem 1.1, we have $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$.

Conversely, suppose that $r_1 \geq 2$ and $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$.

Since $2 \leq h \leq r_0 + r_1 + r_2 - 2$, there exists $(x_0, x_1, x_2) \in R(\mathbf{r}, h)$ such that

$$1 \leq x_0 \leq r_0, \quad 1 \leq x_1 \leq r_1 - 1, \quad 0 \leq x_2 \leq r_2 - 1.$$

Then

$$(x_0, x_1, x_2) \rightarrow (x_0 - 1, x_1 + 1, x_2) \rightarrow (x_0 - 1, x_1, x_2 + 1)$$

and

$$(x_0, x_1, x_2) \rightarrow (x_0, x_1 - 1, x_2 + 1) \rightarrow (x_0 - 1, x_1, x_2 + 1)$$

are two different (\mathbf{r}, h) -paths. By Lemma 2.3, we have

$$\phi_A((x_0 - 1, x_1 + 1, x_2)) = \phi_A((x_0, x_1 - 1, x_2 + 1)).$$

This implies that $a_1 - a_0 = a_2 - a_1$. Therefore, A is a 3-term arithmetic progression. \square

Theorem 3.2. *Let $A = \{a_0 < a_1 < a_2 < a_3\}$ be a set of integers and $\mathbf{r} = (r_0, r_1, r_2, r_3)$ be an ordered 4-tuple of positive integers. Suppose that h is an integer with $2 \leq h \leq r_0 + r_1 + r_2 + r_3 - 2$. Then*

(i) for $r_1 = r_2 = 1$, we have $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$ if and only if $a_1 - a_0 = a_3 - a_2$;

(ii) for $r_1 \geq 2$ or $r_2 \geq 2$, we have $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$ if and only if A is a 4-term arithmetic progression.

Proof. Suppose that $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$.

Since $2 \leq h \leq r_0 + r_1 + r_2 + r_3 - 2$, there exists $(x_0, x_1, x_2, x_3) \in R(\mathbf{r}, h)$ such that

$$1 \leq x_0 \leq r_0, \quad 0 \leq x_1 \leq r_1 - 1, \quad 1 \leq x_2 \leq r_2, \quad 0 \leq x_3 \leq r_3 - 1.$$

Then

$$(x_0, x_1, x_2, x_3) \rightarrow (x_0 - 1, x_1 + 1, x_2, x_3) \rightarrow (x_0 - 1, x_1 + 1, x_2 - 1, x_3 + 1)$$

and

$$(x_0, x_1, x_2, x_3) \rightarrow (x_0, x_1, x_2 - 1, x_3 + 1) \rightarrow (x_0 - 1, x_1 + 1, x_2 - 1, x_3 + 1)$$

are two different (\mathbf{r}, h) -paths. By Lemma 2.3, we have

$$\phi_A((x_0 - 1, x_1 + 1, x_2, x_3)) = \phi_A((x_0, x_1, x_2 - 1, x_3 + 1)).$$

This implies that

$$(6) \quad a_1 - a_0 = a_3 - a_2.$$

We first prove (i).

It is enough to prove that if $r_1 = r_2 = 1$ and $a_1 - a_0 = a_3 - a_2$, then $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$.

By Lemma 2.2, there exists a (\mathbf{r}, h) -path from V to V'

$$(7) \quad V = V_0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_s = V'.$$

Suppose that

$$(8) \quad V = W_0 \rightarrow W_1 \rightarrow \cdots \rightarrow W_t = V'$$

is also a (\mathbf{r}, h) -path from V to V' . By Lemma 2.1, we have $s = t$. Now we prove that $\phi_A(V_i) = \phi_A(W_i)(0 \leq i \leq s)$. In order to prove this, we prove the following stronger result: for $0 \leq i < s$, if $V_i = W_i$, then either $V_{i+1} = W_{i+1}$ or $V_{i+2} = W_{i+2}$ and $\phi_A(V_{i+1}) = \phi_A(W_{i+1})$.

Suppose that $0 \leq i < s$ and $V_i = W_i = (v_{i,0}, v_{i,1}, v_{i,2}, v_{i,3})$.

Case 1: $v_{i,1} = v_{i,2} = 0$. Then, by the definition of step, we have

$$V_{i+1} = (v_{i,0} - 1, v_{i,1} + 1, v_{i,2}, v_{i,3}) = W_{i+1}.$$

Case 2: $v_{i,1} = v_{i,2} = 1$. Then, by the definition of step and $r_1 = r_2 = 1$, we have

$$V_{i+1} = (v_{i,0}, v_{i,1}, v_{i,2} - 1, v_{i,3} + 1) = W_{i+1}.$$

Case 3: $v_{i,1} = 1, v_{i,2} = 0$. Then, by the definition of step and $r_1 = r_2 = 1$, we have

$$V_{i+1} = (v_{i,0}, v_{i,1} - 1, v_{i,2} + 1, v_{i,3} + 1) = W_{i+1}.$$

Case 4: $v_{i,1} = 0, v_{i,2} = 1$. Then, by the definition of step and $r_1 = r_2 = 1$, we have

$$(9) \quad \{V_{i+1}, W_{i+1}\} \subseteq \{(v_{i,0} - 1, v_{i,1} + 1, v_{i,2}, v_{i,3}), (v_{i,0}, v_{i,1}, v_{i,2} - 1, v_{i,3} + 1)\}.$$

Since

$$\begin{aligned} & \phi_A((v_{i,0} - 1, v_{i,1} + 1, v_{i,2}, v_{i,3})) - \phi_A(V_i) \\ &= a_1 - a_0 = a_3 - a_2 \\ &= \phi_A((v_{i,0}, v_{i,1}, v_{i,2} - 1, v_{i,3} + 1)) - \phi_A(V_i), \end{aligned}$$

we have $\phi_A(V_{i+1}) = \phi_A(W_{i+1})$. By (9), the definition of adjacency and $r_1 = r_2 = 1$, we have

$$V_{i+2} = (v_{i,0} - 1, v_{i,1} + 1, v_{i,2} - 1, v_{i,3} + 1) = W_{i+2}.$$

Thus, we have proved that for $0 \leq i < s$, if $V_i = W_i$, then either $V_{i+1} = W_{i+1}$ or $V_{i+2} = W_{i+2}$ and $\phi_A(V_{i+1}) = \phi_A(W_{i+1})$. It follows from $V_0 = W_0$ and $V_s = W_s$ that $\phi_A(V_i) = \phi_A(W_i)(0 \leq i \leq s)$.

For any $W \in R(\mathbf{r}, h)$, by Lemma 2.2, there exists a (\mathbf{r}, h) -path from V to W and a (\mathbf{r}, h) -path W to V' . By the above arguments, we have

$$\phi_A(W) \in \{\phi_A(V_i) : 0 \leq i \leq s\}.$$

Hence

$$h^{(\mathbf{r})}A = \{\phi_A(X) : X \in R(\mathbf{r}, h)\} = \{\phi_A(V_i) : 0 \leq i \leq s\}.$$

Therefore, by Lemma 2.1,

$$|h^{(\mathbf{r})}A| = s + 1 = S(V') - S(V) + 1 = L(\mathbf{r}, h).$$

Now we prove (ii).

If A is a 4-term arithmetic progression, without loss of generality, we may assume that $A = \{0, 1, 2, 3\}$. By the proof of Theorem 1.1, we have $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$.

Conversely, we suppose that $|h^{(\mathbf{r})}A| = L(\mathbf{r}, h)$ and $r_1 \geq 2$ or $r_2 \geq 2$. By (6), it is enough to prove that $a_2 - a_1 = a_1 - a_0$ or $a_2 - a_1 = a_3 - a_2$.

Case 1: $r_1 \geq 2$. Since $2 \leq h \leq r_0 + r_1 + r_2 + r_3 - 2$, there exists $Y = (y_0, y_1, y_2, y_3) \in R(\mathbf{r}, h)$ such that

$$1 \leq y_0 \leq r_0, \quad 1 \leq y_1 \leq r_1 - 1, \quad 0 \leq y_2 \leq r_2 - 1, \quad 0 \leq y_3 \leq r_3.$$

Then

$$(y_0, y_1, y_2, y_3) \rightarrow (y_0 - 1, y_1 + 1, y_2, y_3) \rightarrow (y_0 - 1, y_1, y_2 + 1, y_3)$$

and

$$(y_0, y_1, y_2, y_3) \rightarrow (y_0, y_1 - 1, y_2 + 1, y_3) \rightarrow (y_0 - 1, y_1, y_2 + 1, y_3)$$

are two different (\mathbf{r}, h) -paths. By Lemma 2.3, we have

$$\phi_A((y_0 - 1, y_1 + 1, y_2, y_3)) = \phi_A((y_0, y_1 - 1, y_2 + 1, y_3)).$$

This implies that $a_1 - a_0 = a_2 - a_1$.

Case 2: $r_2 \geq 2$. Since $2 \leq h \leq r_0 + r_1 + r_2 + r_3 - 2$, there exists $Z = (z_0, z_1, z_2, z_3) \in R(\mathbf{r}, h)$ such that

$$0 \leq z_0 \leq r_0, \quad 1 \leq z_1 \leq r_1, \quad 1 \leq z_2 \leq r_2 - 1, \quad 0 \leq z_3 \leq r_3 - 1.$$

Then

$$(z_0, z_1, z_2, z_3) \rightarrow (z_0, z_1 - 1, z_2 + 1, z_3) \rightarrow (z_0, z_1 - 1, z_2, z_3 + 1)$$

and

$$(z_0, z_1, z_2, z_3) \rightarrow (z_0, z_1, z_2 - 1, z_3 + 1) \rightarrow (z_0, z_1 - 1, z_2, z_3 + 1)$$

are two different (\mathbf{r}, h) -paths. By Lemma 2.3, we have

$$\phi_A((z_0, z_1 - 1, z_2 + 1, z_3)) = \phi_A((z_0, z_1, z_2 - 1, z_3 + 1)).$$

This implies that $a_2 - a_1 = a_3 - a_2$.

Therefore, A is a 4-term arithmetic progression. \square

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